

A Geometric Fractal Growth Model for Scale Free Networks

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Abstract

We introduce a deterministic model for scale-free networks, whose degree distribution follows a power-law with an exponent γ . At each time step, each vertex generates its offsprings, whose number is proportional to the degree of that vertex with proportionality constant $m - 1$ ($m > 1$). We consider the two cases: first, each offspring is connected to its parent vertex only, forming a tree structure, and secondly, it is connected to both its parent and grandparent vertices, forming a loop structure. We find that both models exhibit power-law behaviors in their degree distributions with the exponent $\gamma = 1 + \ln(2m - 1)/\ln m$. Thus, by tuning m , the degree exponent can be adjusted in the range, $2 < \gamma < 3$. We also solve analytically a mean shortest-path distance d between two vertices for the tree structure, showing the small-world behavior, that is, $d \sim \ln N / \ln \bar{k}$, where N is system size, and \bar{k} is the mean degree. Finally, we consider the case that the number of offsprings is the same for all vertices, and find that the degree distribution exhibits an exponential-decay behavior.

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I. INTRODUCTION

Recently, complex systems have received considerable attention as an interdisciplinary subject [1,2]. Complex systems consist of many constituents such as individuals, substrates, and companies in social, biological, and economic systems, respectively, showing cooperative phenomena between constituents through diverse interactions and adaptations to the pattern they create [3,4]. Recently, there have been a lot of efforts to understand such complex systems in terms of networks, composed of vertices and edges, where vertices (edges) represent constituents (their interactions). This approach was initiated by Erdős and Rényi (ER) [5]. In the ER model, the number of vertices is fixed, while edges connecting one vertex to another occur randomly with a certain probability. However, the ER model is too random to describe real complex systems. Recently, Barabási and Albert (BA) [6,7] introduced an evolving network where the number of vertices N increases linearly with time rather than fixed, and a newly born vertex is connected to already existing vertices, following the so-called preferential attachment (PA) rule; When the number of edges k incident upon a vertex is called the degree of the vertex, the PA rule means that the probability Π_i for the new vertex to connect to an already existing vertex i is proportional to the degree k_i of the selected vertex, that is,

$$\Pi_i = \frac{k_i}{\sum_j k_j}. \quad (1)$$

The main difference between the ER and BA models appears in the degree distribution. For the ER network, the degree distribution follows the Poisson distribution, while for the BA network, it follows a power-law, $P(k) \sim k^{-\gamma}$ with $\gamma = 3$. The network whose degree distribution follows a power-law is called the scale-free (SF) network [7]. SF networks are abundant in real-world such as the world-wide web [8–11], the Internet [12–15], the citation network [16], the author collaboration network of scientific papers [17], and the metabolic networks in biological organisms [18].

While a lot of models have been introduced to describe SF networks in real world, most of them are stochastic models. However, a couple of models recently introduced by Barabási, Ravasz and Vicsek (BRV) [19], and Dorogovtsev and Mendes (DM) [2] are deterministic. In general, the deterministic model is useful in investigating analytically not only topological features of SF networks in detail, but also dynamical problems on the networks. Both the BRV and the DM models are meaningful since they are not only the first attempts for deterministic SF networks, but also the ones constructed in a hierarchical way, so that analytic treatments can be made easily using recursive relations derived from two structures in successive generations. In the BRV model, however, the mean shortest-path distance between two vertices, called the diameter, is independent of system size. Thus, the model may be relevant to some specific systems such as the metabolic network [18], where the diameter is independent of system size. In this paper, we introduce another type of the deterministic model for the SF network, which is also constructed in a hierarchical way. Our model is based on almost the same idea as that of the DM model. While the DM model starts from a triangle, our model does from a tree structure. This difference makes one easily modify the model into more general cases such as loopless or loop structures, and the ones with a various number of branches. Moreover, the simplicity of our model enables us to obtain the analytic solution for the degree distribution and the diameter. In particular, our model includes a control parameter, so that by tuning the parameter, we can obtain SF networks with a variety of degree exponents in the range, $2 < \gamma < 3$. Therefore our model should be useful to represent various SF networks in real world.

This paper is organized as follows. In section II, we will introduce deterministic models specifically for tree and loop structures, respectively. In section III, analytic treatment will be performed for the deterministic models introduced in section II. The final section will be devoted to conclusions and discussions.

II. DETERMINISTIC MODEL

It is known that the number of vertices in most of SF networks in real world increases exponentially in time. Thus, our deterministic model is constructed in an evolving way, where each already existing vertex produces its offsprings, and the connections are made between old and new vertices. Thus, vertices are generated in a hierarchical order, so that the number of vertices increases geometrically in time. On the other hand, it is known [20,21] that the PA probability Π'_i , Eq.(1) is generalized for real networks as

$$\Pi'_i = \frac{k_i + \mu}{\sum_j k_j + \mu}, \quad (2)$$

where μ accounts for some randomness in connecting edges. To take into account of this modified PA behavior, we introduce two rules, called the addition and the multiplication rule, in the deterministic model, depending on how new vertices are generated from each old vertex. The details on both rules will be described below.

A. Tree structure

The network forms a tree structure when new vertices generated from an old vertex are connected to their parent only.

1. The addition rule

In the case of the addition rule, at each time step, a constant number of new vertices, say ℓ new vertices, are generated from each already existing vertex, and they are connected to their parent only. Then the degree $k_{i,a}$ at vertex i , where the subscript a means the addition rule, evolves as

$$k_{i,a}(t+1) = k_{i,a}(t) + \ell, \quad (3)$$

so that

$$k_{i,a}(t) = 1 + \ell(t - t_i), \quad (4)$$

for $t \geq t_i$, where t_i means the time when the vertex i was born. Then the total number of vertices newly born at time t becomes $\mathcal{L}_a(t) = \ell(1 + \ell)^t$ for $t \geq 1$, and the total number of vertices $N_a(t)$ present at time t is

$$N_a(t) = \sum_{j=0}^t \mathcal{L}_a(j) = (1 + \ell)^{t+1}, \quad (5)$$

where $N_a(0) = 1 + \ell$ is chosen. The definition of this model is illustrated schematically in Fig.1.

2. The multiplication rule

In the case of the multiplication rule, the number of offsprings generated from each old vertex is not the same, but it depends on the degree of each vertex. Let $k_{i,m}(t)$ be the degree of vertex i at time t , where the subscript m means the multiplication rule. Then the number of offsprings generated at time $t + 1$ from the vertex i is proportional to its degree at the previous time, i.e., $(m - 1)k_{i,m}(t)$, where $m - 1$ is a proportionality constant. Thus the degree of vertex i increases by a factor m for each time step, that is,

$$k_{i,m}(t) = mk_{i,m}(t - 1). \quad (6)$$

Thus, the degree of vertex i at time t is

$$k_{i,m} = m^{t-t_i}, \quad (7)$$

for $t \geq t_i$. The total number of vertices newly born at time t , $\mathcal{L}_m(t)$ can be obtained to be

$$\begin{aligned} \mathcal{L}_m(t) &= 2(m - 1)m \sum_{p=0}^{t-1} \binom{t-1}{p} m^p (m - 1)^{t-1-p}, \\ &= 2m(m - 1)(2m - 1)^{t-1}, \end{aligned} \quad (8)$$

for $t \geq 1$. The total number of vertices $N_m(t)$ at time t is given by

$$N_m(t) = \sum_{j=0}^t \mathcal{L}_m(j) = 1 + m(2m - 1)^t. \quad (9)$$

The definition of this model is illustrated schematically in Fig.2.

One may write the rate equation for the degree in this multiplicative process with continuous time as

$$\frac{\partial k_{i,m}}{\partial t} = (m-1)k_{i,m}. \quad (10)$$

It would be interesting to compare this rate equation with the one for the preferential attachment (PA), in which the degree k_i of vertex i evolves as

$$\frac{\partial k_i}{\partial t} = \mathcal{L}_m \frac{k_i}{\sum_j k_j}, \quad (11)$$

where $\mathcal{L}_m(t)$ means the total number of edges newly introduced at time t . Since the total number of the degree at time t is given by

$$\sum_j k_j = 2m(2m-1)^{t-1}, \quad (12)$$

and \mathcal{L}_m is given by Eq.(8), Eq.(11) is reduced to Eq.(10), indicating that the rate equation for the degree in the multiplicative process is equivalent to the one for the preferential attachment.

B. Loop structure

The loop structure can be formed in networks, when a newly born vertex is connected to more than one existing vertices. For the loop structure, each already existing vertex generates the same number of offsprings as those for the tree structure. However, a newly born vertex is connected to two distinct old vertices: one is its parent, and the other is its grandparent. When the parent is one of vertices on m branches (the centered one) born at $t=0$, the centered one (one of vertices on m branches in a symmetrical way) is regarded as a grandparent. This rule is valid for both cases of the addition rule and the multiplication rule. The details on the connection rule is illustrated in Fig.3.

III. ANALYTIC SOLUTION

A. The degree distribution for the tree structure

Since the degree of a vertex has been obtained explicitly as in Eqs.(4) and (7) and they are ordered with time, we can obtain the degree distribution using the relation,

$$P(k) = \frac{\partial[1 - P(k_i(t) > k)]}{\partial k}, \quad (13)$$

which is valid for both cases of the addition and the multiplication rules. The detail of analytic treatments for the degree distributions for both cases are given as follows.

1. The addition rule

Using the fact, $P_{a,t}(k_{i,a} > k) = P_{a,t}(t_i < t - (k - 1)/\ell)$, where the subscript t means the tree structure, we obtain that

$$\begin{aligned} P_{a,t}(k_{i,a}(t) > k) &= \frac{\ell}{(1 + \ell)^{t+1}} \sum_{t_i=0}^{t-(k-1)/\ell} (1 + \ell)^{t_i} \\ &= (1 + \ell)^{-(k-1)/\ell} - (1 + \ell)^{-(t+1)}. \end{aligned} \quad (14)$$

Applying Eq.(13) to Eq.(14), we obtain the degree distribution to be

$$P_{a,t}(k) \propto (1 + \ell)^{-(k/\ell)}. \quad (15)$$

So, the degree distribution $P_{a,t}(k)$ in the addition rule decays exponentially with k .

2. The multiplication rule

Since the degree k_i has been obtained explicitly as a function of time in Eq.(7), $P_{m,t}(k_i > k)$ is written as $P_{m,t}(k_i > k) = P_{m,t}(t_i < \tau)$, where $\tau = t - \ln k / \ln m$. Since $P_{m,t}(t_i < \tau)$ means the density of the vertices born earlier than τ ,

$$\begin{aligned}
P_{m,t}(k_i > k) &= \sum_{t_i=0}^{\tau} \frac{\mathcal{L}_m(t_i)}{N_m(t)} \\
&= \sum_{t_i=0}^{\tau} \frac{2(m-1)(2m-1)^{t_i-1}}{1+m(2m-1)^t} \\
&\propto k^{-\ln(2m-1)/\ln m}.
\end{aligned} \tag{16}$$

Thus the degree distribution is obtained to be

$$\begin{aligned}
P_{m,t}(k) &= \frac{\partial[1 - P_{m,t}(k_i(t) > k)]}{\partial k} \\
&\propto k^{-\gamma(m)},
\end{aligned} \tag{17}$$

where

$$\gamma(m) = 1 + \ln(2m-1)/\ln m. \tag{18}$$

In the limit of $m \rightarrow 1$, we get $\gamma(1) = 3$, while as m goes to infinity, we get $\gamma(\infty) = 2$. Thus by tuning the parameter m , we can get a variety of SF networks with different exponents in the range, $2 < \gamma < 3$.

B. The degree distribution for the loop structure

1. The addition rule

Let $n_{i,a}(t)$ be the degree of vertex i at time t for the loop structure, where a means the addition rule. Each old vertex receives edges from its ℓ children and ℓ^2 grandchildren as they are born. So, Eq.(3) is modified as

$$n_{i,a}(t+1) = n_{i,a}(t) + (\ell + \ell^2). \tag{19}$$

Thus, the degree distribution shows an exponential-decay behavior,

$$P_{a,l}(n) \propto (1 + \ell + \ell^2)^{-n/(\ell + \ell^2)}, \tag{20}$$

where the subscript l means the loop structure.

2. The multiplication rule

Let $n_{i,m}(t)$ be the degree of vertex i at time t in the multiplication rule for the loop structure. The degree of vertex i can be obtained,

$$\begin{aligned} n_{i,m}(t) &= n_{i,m}(t-1) + (m-1)k_{i,m}(t-1) \\ &\quad + (m-1)^2 k_{i,m}(t-2), \end{aligned} \quad (21)$$

where the second term on the right hand side of the above equation results from the children of the vertex i , and the third term from the grandchildren of the vertex i . Thus, the degree at the vertex i becomes

$$n_{i,m}(t) = 2m^{t-t_i} - m^{t-t_i-1} - m \approx \left(\frac{2m-1}{m}\right)m^{t-t_i}. \quad (22)$$

Since the degree $n_{i,m}(t)$ depends on time t similarly to Eq.(7), we can apply Eq.(16) even to the loop case, except that τ is replaced by $\tau = t + \ln(2m-1)/\ln m - 1 - \ln n/\ln m$. This replacement, however, does not affect the degree exponent at all. Thus, even for the loop structure, the degree exponent is reduced to the same value, $\gamma = 1 + \ln(2m-1)/\ln m$ in Eq.(18).

C. The diameter for the tree structure

The diameter $d(t)$ is defined as a geodesic distance between two distinct vertices along the shortest path averaged over all pairs of vertices at time t , that is,

$$d(t) = \frac{1}{N(t)(N(t)-1)} \sum_{i \neq j} d_{i,j}(t), \quad (23)$$

where $d_{i,j}(t)$ is the shortest-path distance between vertex i to j . For simplicity, let $\mathcal{D}(t)$ denote the sum of the shortest-path distances between two vertices over all pairs, that is,

$$\mathcal{D}(t) = \sum_{i \neq j} d_{i,j}(t). \quad (24)$$

It is not easy to obtain a closed formula for $\mathcal{D}(t)$ for both the tree and the loop structure, however, we list $\mathcal{D}(t)$ for the tree structure in a few early times in Appendix. We trace the

formula for the tree structure in two limiting cases, $m \rightarrow 0$ and $m \rightarrow \infty$, as follows.

Let us first consider the case of $m \rightarrow 1$. For this case, we denote $m = 1 + \epsilon$ and $\epsilon \ll 1$. The total number of nodes $N(t)$ at time t is given by

$$\begin{aligned} N(t) &= 1 + (1 + 2\epsilon)^t(1 + \epsilon) \\ &\approx 2 + (2t + 1)\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \quad (25)$$

Moreover, the sum of all shortest-path distances $\mathcal{D}(t)$ becomes

$$\mathcal{D}(t) \approx 2 + 4(2t + 1)\epsilon + \mathcal{O}(\epsilon^2). \quad (26)$$

Using the relation Eqs.(23), we can obtain the average distance to be

$$\begin{aligned} d &= \frac{2 + 4(2t + 1)\epsilon}{2 + 3(2t + 1)\epsilon} + \mathcal{O}(\epsilon^2), \\ &\cong \frac{-8}{7 + 6 \log(N - 1)} + \frac{4}{3}. \end{aligned} \quad (27)$$

Therefore, the diameter converges to $4/3$ in the limit of $N \rightarrow \infty$.

Next, we consider the case of $m \rightarrow \infty$. In this case, the term in the highest order of m could be dominant, so that we trace the coefficient of the term in the highest order of m as a function of time.

$$\begin{aligned} \mathcal{D}(0) &= 2m^2 + \text{lower order terms} \\ \mathcal{D}(1) &= [(2 + 3) + (3 + 4)]m^4 + \text{lower order terms} \\ \mathcal{D}(2) &= [(2 + 2 \cdot 3 + 4) + 2 \cdot (3 + 2 \cdot 4 + 5) + (4 + 2 \cdot 5 + 6)]m^6 \\ &\quad + \text{lower order terms} \\ \mathcal{D}(3) &= [(2 + 3 \cdot 3 + 3 \cdot 4 + 5) + 3(3 + 3 \cdot 4 + 3 \cdot 5 + 6) \\ &\quad 3(4 + 3 \cdot 5 + 3 \cdot 6 + 7) + (5 + 3 \cdot 6 + 3 \cdot 7 + 8)]m^8 \\ &\quad + \text{lower order terms} \end{aligned}$$

$$\begin{aligned}
& \vdots \\
\mathcal{D}(t) &= \sum_{p=0}^t \binom{t}{p} \sum_{k=0}^t \binom{t}{k} (k+p+2) m^{2(t+1)} \\
& \quad + \text{lower order terms.}
\end{aligned} \tag{28}$$

Therefore the term in the highest order of m for $\mathcal{D}(t)$ is obtained explicitly to be

$$\mathcal{D}(t) \approx (t+2)2^{2t}m^{2(t+1)}, \tag{29}$$

where the coefficient $(t+2)2^{2t}$ means the number of pathways having the distance $2(t+1)$, which is the farthest one at time t in the system. On the other hand,

$$N(t)(N(t)-1) \approx 2^{2t}m^{2(t+1)}. \tag{30}$$

Therefore, the diameter $d(t)$ at time t becomes simply

$$\begin{aligned}
d(t) &\approx t+2 \\
&\approx \frac{\log(N-1)}{\log(2m-1)} + 2.
\end{aligned} \tag{31}$$

Thus, for large N , the above equation is rewritten simply as

$$d(N) \sim \ln N / \ln \bar{k} \tag{32}$$

with the mean degree $\bar{k} \approx 2m$, which confirms the small-world behavior.

IV. CONCLUSIONS AND DISCUSSIONS

We have introduced a deterministic model for the scale-free network, which is constructed in a hierarchical way. At each time step, each already existing vertex produces its offsprings, whose number is proportional to the degree of the vertex. Depending on whether each offspring is connected to only one or more than one old vertices, the network forms either a tree structure or a loop structure, respectively. We have obtained the analytic solution for the degree distribution and the diameter explicitly for the deterministic model. By tuning a control parameter in the model, we can adjust the degree exponent in the range, $2 < \gamma < 3$.

Thus this model can represent a variety of SF networks in real world. Moreover, we obtained the diameter of the deterministic model analytically to be $d \sim \ln N / \ln \bar{k}$, where N is the system size and \bar{k} is the mean degree. Since the network is generated in a hierarchical way, it is expected that a variety of physical problems can be solved through this deterministic model by constructing recursive relations derived from two structures in successive generations. On the other hand, the deterministic model has a shortcoming that it does not include any long-ranged edge, connecting two vertices belonging to different branches separated at $t = 0$. Thus, while this model can be easily generalized by controlling the number of branches m , it is extremely vulnerable, and can be broken into pieces by a simple deletion of the centered vertex. Despite this shortcoming, we think that our deterministic model could offer a guide toward generating more realistic deterministic model for SF networks.

V. ACKNOWLEDGMENT

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VI. APPENDIX

The closed formula for the sum of the shortest-path distance between two vertices, $\mathcal{D}(t)$ are shown for $0 \leq t \leq 2$.

$$\mathcal{D}(t=0) = \mathcal{N}_0 m + \mathcal{N}_{0,0}[1 + 2(m-1)],$$

$$\begin{aligned} \mathcal{D}(t=1) = & \mathcal{N}_0[m^2 + 2m(m-1)] + \mathcal{N}_{0,0}[m + 2m(m-1) + 2(m-1) + 3(m-1)(m-1)] \\ & + \mathcal{N}_{1,0}[1 + 2m^2 - 2(m+1) + 2m + 3m(m-1)] \\ & + \mathcal{N}_{1,0,0}[1 + 2(m-1) + 3(m^2-1) + 4(m-1)^2], \end{aligned}$$

$$\begin{aligned} \mathcal{D}(t=2) = & \mathcal{N}_0[m^3 + 2(m^2-m)(m-1) + 2m(m-1) + 2m(m^2-m) + 3m(m-1)(m-1)] \\ & + \mathcal{N}_{0,0}[1m^2 + 2m^3 - 2 + 2(m-1)(m-1) + 3\{(m-1)^2 + (m^2-m)(m-1)\}] \end{aligned}$$

$$\begin{aligned}
& + (m-1)(m^2-m)\} + 4(m-1)^3] \\
& + \mathcal{N}_{1,0}[m + 2(m^3-1) + 3\{(m(m-1)-1)(m-1) + m(m^2-m) + m(m-1)\} \\
& + 4m(m-1)^2] \\
& + \mathcal{N}_{1,0,0}[m + 2(m^2-1) + 3\{(m^3-1) + (m-1)(m-2)\} \\
& + 4\{(m^2-m)(m-1) + (m-1)^2 + (m-1)(m^2-m)\} + 5(m-1)^3] \\
& + \mathcal{N}_{2,0}[1 + 2(m^3-1) + 3\{m^2(m-1) + m(m^2-m)\} + 4m(m-1)^2] \\
& + \mathcal{N}_{2,1,0}[1 + 2(m-1) + 3(m^3-1) + 4\{(m(m-1)-1)(m-1) + m(m^2-1)\} \\
& + 5m(m-1)^2] \\
& + \mathcal{N}_{2,0,0}[1 + 2(m^2-1) + 3\{(m^3-1) + (m-1)(m-1)\} \\
& + 4\{(m^2-1)(m-1) + (m-1)(m^2-m)\} + 5(m-1)^3] \\
& + \mathcal{N}_{2,1,0,0}[1 + 2(m-1) + 3(m^2-1) + 4\{(m^3-1) + (m-1)(m-1-1)\} \\
& + 5\{(m^2-1)(m-1) + (m-1)(m^2-1)\} + 6(m-1)^3], \\
& \vdots
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{N}_0 &= 1, \\
\mathcal{N}_{0,0} &= m, \\
\mathcal{N}_{1,0} &= m^2 - m, \\
\mathcal{N}_{1,0,0} &= m(m-1), \\
\mathcal{N}_{2,0} &= m^3 - m^2, \\
\mathcal{N}_{2,1,0} &= (m^2 - m)(m-1), \\
\mathcal{N}_{2,0,0} &= m(m^2 - m),
\end{aligned}$$

and

$$\mathcal{N}_{2,1,0,0} = m(m-1)(m-1).$$

$\mathcal{N}_{i,j}$ means the number of the vertices denoted by $A_{i,j}$ in Fig.2, where the first index i stands for its birth time and the rest indices j do its parent vertex.

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FIGURES

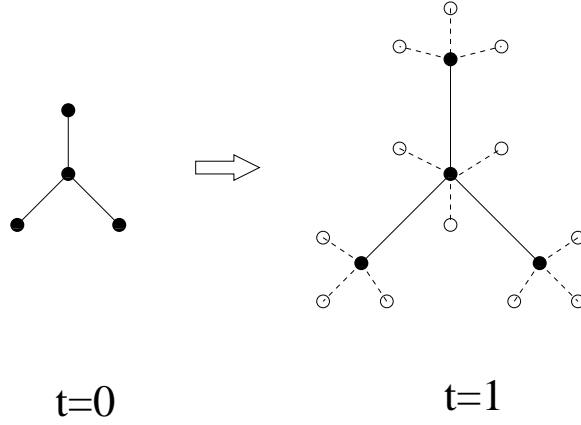


FIG. 1. Tree structures in the addition rule with $\ell = 3$ at $t = 0$ and $t = 1$.

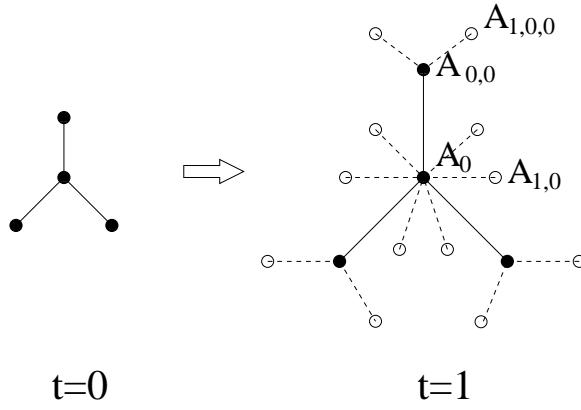


FIG. 2. Tree structures in the multiplication rule with $m = 3$ at $t = 0$ and $t = 1$. A_0 stands for the vertex at center, $A_{0,0}$, a neighbor of A_0 born at $t = 0$, and $A_{1,0}$ ($A_{1,0,0}$), an offspring of A_0 ($A_{0,0}$) born at $t = 1$.

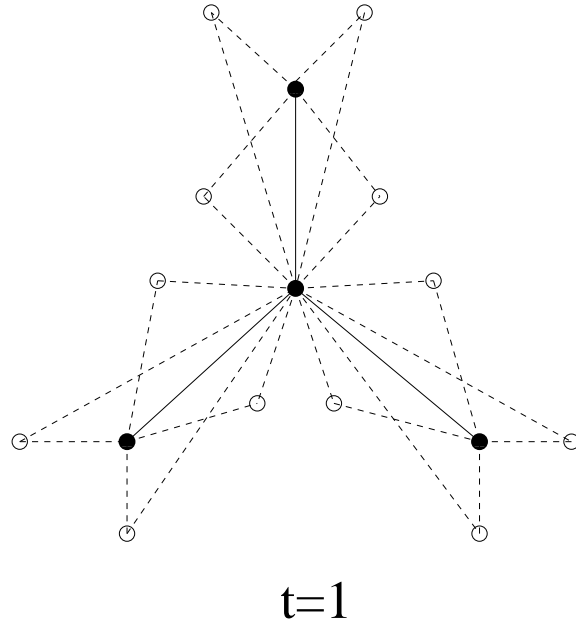


FIG. 3. Loop structure in the multiplication rule with $m = 3$ at $t = 1$.